Dynamic stochastic general equilibrium inference using a score-driven approach

Szabolcs Blazsek (Universidad Francisco Marroquin)

Alvaro Escribano (Universidad Carlos III de Madrid)

Adrian Licht (Universidad Francisco Marroquin)

Structure:

1) **Score-driven DSGE models**
2) **An and Schorfheide (2007) – "AS model"**
3) **Data**
4) **Results**

Score-driven DSGE models

DSGE (dynamic stochastic general equilibrium)

- **DSGE models** are nonlinear systems of equations which, when loglinearized around the steady state, can be represented by a **system of linear expectational difference equations** (Iskrev 2010; Kociecki and Kolasa 2018).
- We focus on DSGE models, for which the **minimal state-space solution of the system of expectational difference equations exists** (Komunjer and Ng 2011); this is the case for most of the DSGE models (Morris 2014).
- The minimal state-space solution has a **Gaussian ABCD representation** (Fernandez-Villaverde et al. 2007; Komunjer and Ng 2011), which is a commonly used representation of DSGE models in the literature (Giacomini 2013).

- **Score-driven models:**
- Harvey and Chakravarty (2008); Creal, Koopman, and Lucas (2008)
- Until now, more that 200 publications in academic journals on score-driven models. One of the most important advances in time series econometrics of the last decade.
- **Advantages of score-driven models:**
- Generalizations of many classical time series models (for example, ARMA, GARCH, VAR).
- From an information-theoretic point of view, optimal updating mechanism.
- Robustness to outliers and missing observations in the time series.

1) Gaussian ABCD representation

$$Y_t = C(\Theta)X_{t-1} + D(\Theta)\epsilon_t = C(\Theta)X_{t-1} + v_t$$

$$X_t = A(\Theta)X_{t-1} + B(\Theta)\epsilon_t = A(\Theta)X_{t-1} + B(\Theta)D^{-1}(\Theta)v_t$$

- where $\varepsilon_t \sim N_K(0_{K\times 1}, \Sigma)$ has a multivariate normal distribution with variance-covariance matrix Σ.
- Estimated by using the **maximum likelihood (ML)** method. **Impulse response functions** are available, since the model has a Gaussian-VARMA(1,1) representation.

2) Score-driven homoscedastic ABCD representation

$$Y_t = C(\Theta)X_{t-1} + D(\Theta)\epsilon_t = C(\Theta)X_{t-1} + v_t$$

$$X_t = A(\Theta)X_{t-1} + B(\Theta)D^{-1}(\Theta)u_t$$

- where $\varepsilon_t \sim t_K(0_{K\times 1}, \Omega\Omega', \nu)$ has a multivariate *t*-distribution with scale matrix $\Omega\Omega'$, and degrees of freedom $\nu > 2$.
- Hence, $v_t \sim t_K(0_{K\times 1}, D(\Theta)\Omega\Omega' D(\Theta)', \nu) \equiv t_K(0_{K\times 1}, \Sigma, \nu)$
- **u_t (score function) replaces v_t in the transition equation.**

2) Score-driven homoscedastic ABCD representation

- Score function:
- The conditional mean $\mathrm{E}(Y_t|Y_1, \ldots, Y_{t-1}) = C(\Theta)X_{t-1}$.
- Score function u_t is defined as follows:

$$\frac{\partial \ln f(Y_t|Y_1,\ldots,Y_{t-1})}{\partial[C(\Theta)X_{t-1}]} = \frac{\nu+K}{\nu}\Sigma^{-1} \times \boxed{\left(1+\frac{v_t'\Sigma^{-1}v_t}{\nu}\right)^{-1} v_t} = \frac{\nu+K}{\nu}\Sigma^{-1} \times u_t$$

- u_t is i.i.d. with zero mean vector and a well-defined variance-covariance matrix.

2) Score-driven homoscedastic ABCD representation

- The **Gaussian ABCD representation** is a special case of the **score-driven homoscedastic ABCD representation**, because $u_t \to_p v_t$ as $\nu \to \infty$.
- The score-driven homoscedastic ABCD representation is estimated by using the **maximum likelihood (ML)** method.
- **Impulse response functions** are reported in the paper.

3) Score-driven heteroscedastic ABCD representation

$$Y_t = C(\Theta)X_{t-1} + D(\Theta)\epsilon_t = C(\Theta)X_{t-1} + v_t$$

$$X_t = A(\Theta)X_{t-1} + B(\Theta)D^{-1}(\Theta)u_t$$

- where $\varepsilon_t \sim t_K(0_{K\times 1}, \Omega_t\Omega_t', \nu)$. Hence, $v_t \sim t_K(0_{K\times 1}, D(\Theta)\Omega_t\Omega_t' D(\Theta)', \nu) \equiv t_K(0_{K\times 1}, \Sigma_t, \nu)$
- Estimated by using the **maximum likelihood (ML)** method. **Impulse response functions** are reported in the paper.

3) Score-driven heteroscedastic ABCD representation

- Score function:
- In order to define the score function, in the equation:

$$\frac{\partial \ln f(Y_t|Y_1,\ldots,Y_{t-1})}{\partial[C(\Theta)X_{t-1}]} = \frac{\nu+K}{\nu}\Sigma^{-1} \times \left(1+\frac{v_t'\Sigma^{-1}v_t}{\nu}\right)^{-1} v_t = \frac{\nu+K}{\nu}\Sigma^{-1} \times u_t$$

- Σ is replaced by Σ_t.
- For the model of Σ_t, we use a score-driven specification of $\Omega_t\Omega_t'$ (we show this for a specific DSGE model in the following).

DSGE model of *An and Schorfheide (2007)*

"AS model"

The variables in the AS model are defined as follows:

- y_t is the difference between current output and steady state output.
- π_t is the difference between current inflation and steady state inflation.
- r_t is the difference between current interest rate and steady state interest rate.
- g_t is the difference between current government spending and steady state government spending.
- z_t is the error term in $\ln A_t = \gamma + \ln A_{t-1} + z_t$, where A_t is aggregate productivity.
- c_t is the difference between current consumption and steady state consumption.

$$y_t = E_t(y_{t+1}) + g_t - E_t(g_{t+1}) - \frac{1}{\tau}\left[r_t - E_t(\pi_{t+1}) - E_t(z_{t+1})\right]$$

$$\pi_t = \zeta E_t(\pi_{t+1}) + \frac{\tau(1-\xi)}{\xi\overline{\pi}^2\phi}(y_t - g_t) = \zeta E_t(\pi_{t+1}) + \kappa(y_t - g_t)$$

$$r_t = \rho_r r_{t-1} + (1-\rho_r)\psi_1\pi_t + (1-\rho_r)\psi_2(y_t - g_t) + \epsilon_{r,t}$$

$$g_t = \rho_g g_{t-1} + \epsilon_{g,t}$$

$$z_t = \rho_z z_{t-1} + \epsilon_{z,t}$$

$$c_t = y_t - g_t$$

1) Gaussian ABCD representation (measurment & transition equations)

$$\underbrace{\begin{bmatrix} r_t \\ y_t \\ \pi_t \end{bmatrix}}_{Y_t} = \underbrace{\begin{bmatrix} c_{r,z} & 0 & c_{r,r} \\ c_{y,z} & \rho_g & c_{y,r} \\ c_{\pi,z} & 0 & c_{\pi,r} \end{bmatrix}}_{C(\Theta)} \underbrace{\begin{bmatrix} z_{t-1} \\ g_{t-1} \\ r_{t-1} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} c_{r,z}/\rho_z & 0 & c_{r,r}/\rho_r \\ c_{y,z}/\rho_z & 1 & c_{y,r}/\rho_r \\ c_{\pi,z}/\rho_z & 0 & c_{\pi,r}/\rho_r \end{bmatrix}}_{D(\Theta)} \underbrace{\begin{bmatrix} \epsilon_{z,t} \\ \epsilon_{g,t} \\ \epsilon_{r,t} \end{bmatrix}}_{\epsilon_t}$$

$$\underbrace{\begin{bmatrix} z_t \\ g_t \\ r_t \end{bmatrix}}_{X_t} = \underbrace{\begin{bmatrix} \rho_z & 0 & 0 \\ 0 & \rho_g & 0 \\ c_{r,z} & 0 & c_{r,r} \end{bmatrix}}_{A(\Theta)} \underbrace{\begin{bmatrix} z_{t-1} \\ g_{t-1} \\ r_{t-1} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c_{r,z}/\rho_z & 0 & c_{r,r}/\rho_r \end{bmatrix}}_{B(\Theta)} \underbrace{\begin{bmatrix} \epsilon_{z,t} \\ \epsilon_{g,t} \\ \epsilon_{r,t} \end{bmatrix}}_{\epsilon_t}$$

2) Score-driven homoscedastic ABCD representation

$$\underbrace{\begin{bmatrix} r_t \\ y_t \\ \pi_t \end{bmatrix}}_{Y_t} = \underbrace{\begin{bmatrix} c_{r,z} & 0 & c_{r,r} \\ c_{y,z} & \rho_g & c_{y,r} \\ c_{\pi,z} & 0 & c_{\pi,r} \end{bmatrix}}_{C(\Theta)} \underbrace{\begin{bmatrix} z_{t-1} \\ g_{t-1} \\ r_{t-1} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} c_{r,z}/\rho_z & 0 & c_{r,r}/\rho_r \\ c_{y,z}/\rho_z & 1 & c_{y,r}/\rho_r \\ c_{\pi,z}/\rho_z & 0 & c_{\pi,r}/\rho_r \end{bmatrix}}_{D(\Theta)} \underbrace{\begin{bmatrix} \epsilon_{z,t} \\ \epsilon_{g,t} \\ \epsilon_{r,t} \end{bmatrix}}_{\epsilon_t}$$

$$\underbrace{\begin{bmatrix} z_t \\ g_t \\ r_t \end{bmatrix}}_{X_t} = \underbrace{\begin{bmatrix} \rho_z & 0 & 0 \\ 0 & \rho_g & 0 \\ c_{r,z} & 0 & c_{r,r} \end{bmatrix}}_{A(\Theta)} \underbrace{\begin{bmatrix} z_{t-1} \\ g_{t-1} \\ r_{t-1} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c_{r,z}/\rho_z & 0 & c_{r,r}/\rho_r \end{bmatrix}}_{B(\Theta)} D^{-1}(\Theta) \underbrace{\begin{bmatrix} u_{z,t} \\ u_{g,t} \\ u_{r,t} \end{bmatrix}}_{u_t}$$

3) Score-driven heteroscedastic ABCD representation (the same form)

$$\underbrace{\begin{bmatrix} r_t \\ y_t \\ \pi_t \end{bmatrix}}_{Y_t} = \underbrace{\begin{bmatrix} c_{r,z} & 0 & c_{r,r} \\ c_{y,z} & \rho_g & c_{y,r} \\ c_{\pi,z} & 0 & c_{\pi,r} \end{bmatrix}}_{C(\Theta)} \underbrace{\begin{bmatrix} z_{t-1} \\ g_{t-1} \\ r_{t-1} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} c_{r,z}/\rho_z & 0 & c_{r,r}/\rho_r \\ c_{y,z}/\rho_z & 1 & c_{y,r}/\rho_r \\ c_{\pi,z}/\rho_z & 0 & c_{\pi,r}/\rho_r \end{bmatrix}}_{D(\Theta)} \underbrace{\begin{bmatrix} \epsilon_{z,t} \\ \epsilon_{g,t} \\ \epsilon_{r,t} \end{bmatrix}}_{\epsilon_t}$$

$$\underbrace{\begin{bmatrix} z_t \\ g_t \\ r_t \end{bmatrix}}_{X_t} = \underbrace{\begin{bmatrix} \rho_z & 0 & 0 \\ 0 & \rho_g & 0 \\ c_{r,z} & 0 & c_{r,r} \end{bmatrix}}_{A(\Theta)} \underbrace{\begin{bmatrix} z_{t-1} \\ g_{t-1} \\ r_{t-1} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c_{r,z}/\rho_z & 0 & c_{r,r}/\rho_r \end{bmatrix}}_{B(\Theta)} D^{-1}(\Theta) \underbrace{\begin{bmatrix} u_{z,t} \\ u_{g,t} \\ u_{r,t} \end{bmatrix}}_{u_t}$$

3) Score-driven heteroscedastic ABCD representation

$$\Omega_t\Omega_t' = \begin{bmatrix} \exp(2\lambda_{z,t}) & 0 & 0 \\ 0 & \exp(2\lambda_{g,t}) & 0 \\ 0 & 0 & \exp(2\lambda_{r,t}) \end{bmatrix}$$

$$\lambda_{z,t} = \omega_z + \beta_z\lambda_{z,t-1} + \alpha_z e_{z,t-1} + \alpha_z^*\text{sgn}(-\epsilon_{z,t-1})(e_{z,t-1} + 1)$$

$$\lambda_{g,t} = \omega_g + \beta_g\lambda_{g,t-1} + \alpha_g e_{g,t-1} + \alpha_g^*\text{sgn}(-\epsilon_{g,t-1})(e_{g,t-1} + 1)$$

$$\lambda_{r,t} = \omega_r + \beta_r\lambda_{r,t-1} + \alpha_r e_{r,t-1} + \alpha_r^*\text{sgn}(-\epsilon_{r,t-1})(e_{r,t-1} + 1)$$

3) Score-driven heteroscedastic ABCD representation

The partial derivatives of the log densities of z_t, g_t, and r_t, which are univariate Student's t-distributions, with respect to $\lambda_{z,t}$, $\lambda_{g,t}$, and $\lambda_{r,t}$ are the score functions with respect to log-scale, and are given by:

$$e_{z,t} = [(\nu + 1)v_{z,t}^2]/[\nu \exp(2\lambda_{z,t}) + v_{z,t}^2] - 1$$

$$e_{g,t} = [(\nu + 1)v_{g,t}^2]/[\nu \exp(2\lambda_{g,t}) + v_{g,t}^2] - 1$$

$$e_{r,t} = [(\nu + 1)v_{r,t}^2]/[\nu \exp(2\lambda_{r,t}) + v_{r,t}^2] - 1$$

This is a Beta-t-EGARCH type specification (Harvey and Charkravarty 2008). We also refer to: Angelini and Gorgi (2018).

Data

Data

- Data for the following variables are from Federal Reserve Economic Data (FRED) for the period of 1954 Q3 to 2019 Q4:
- (a) not seasonally adjusted effective federal funds rate $\rightarrow r_t$
- (b) seasonally adjusted US GDP level $\rightarrow y_t$
- (c) seasonally adjusted US CPI for all urban consumers $\rightarrow \pi_t$
- The observation period is the maximum period for which data are available from FRED for these variables.
- Steady states are estimated by using sample averages.

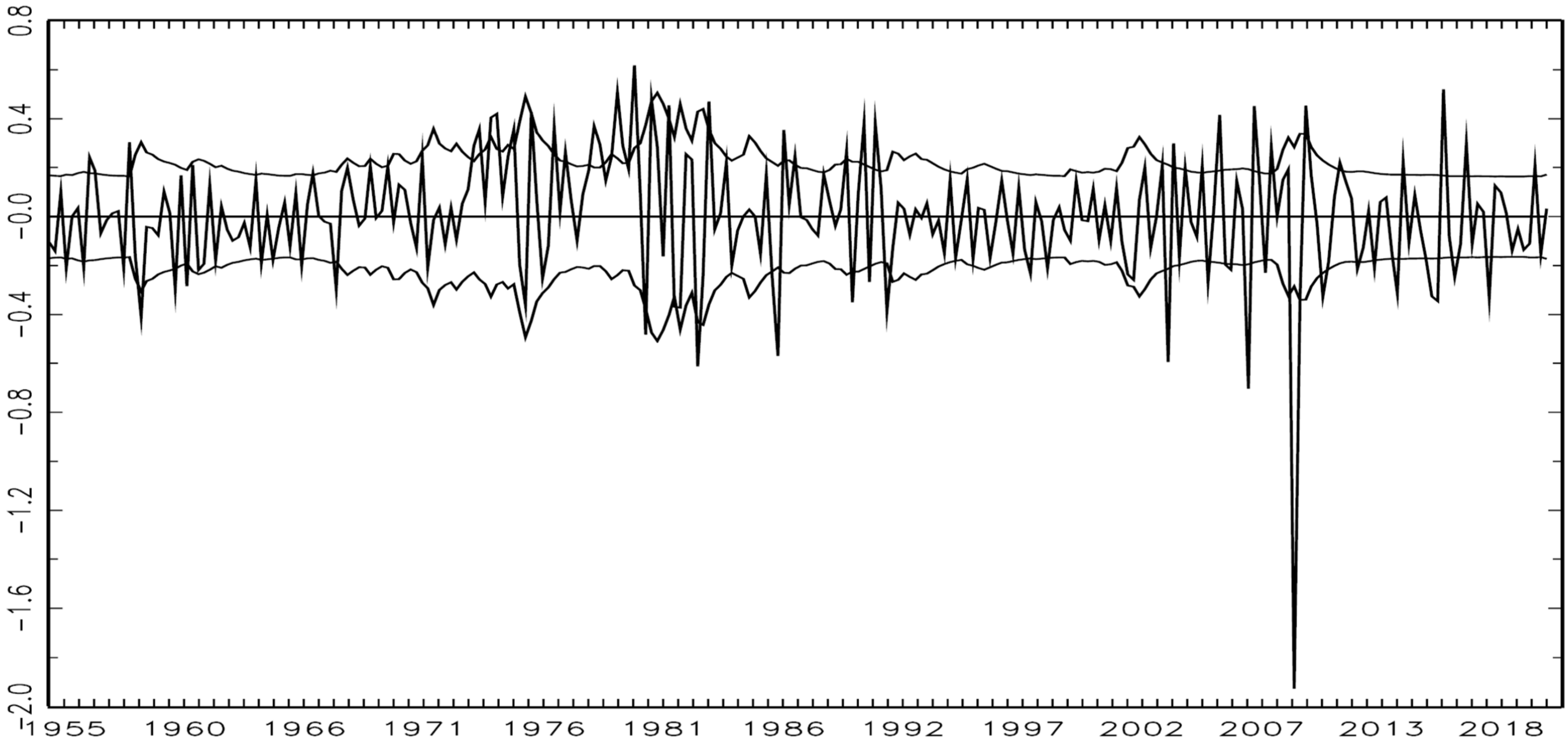
Results

Likelihood-based statistical performance
Statistics of covariance stationarity and invertibility

Gaussian ABCD		Score-driven homoskedastic ABCD		Score-driven heteroskedastic ABCD	
LL	-3.0572	LL	-2.9831	LL	-2.5570
AIC	6.2289	AIC	6.0882	AIC	5.3048
BIC	6.4332	BIC	6.3062	BIC	5.6453
HQC	6.3110	HQC	6.1758	HQC	5.4416
Stat_μ	0.9694	Stat_μ	0.9671	Stat_μ	0.9583
				$\text{Stat}_{\lambda,z}$	0.7668
				$\text{Stat}_{\lambda,g}$	0.9025
				$\text{Stat}_{\lambda,r}$	0.9081
				$\text{Inv}_{\lambda,z}$	-0.3950
				$\text{Inv}_{\lambda,g}$	-0.3066
				$\text{Inv}_{\lambda,r}$	-0.0813

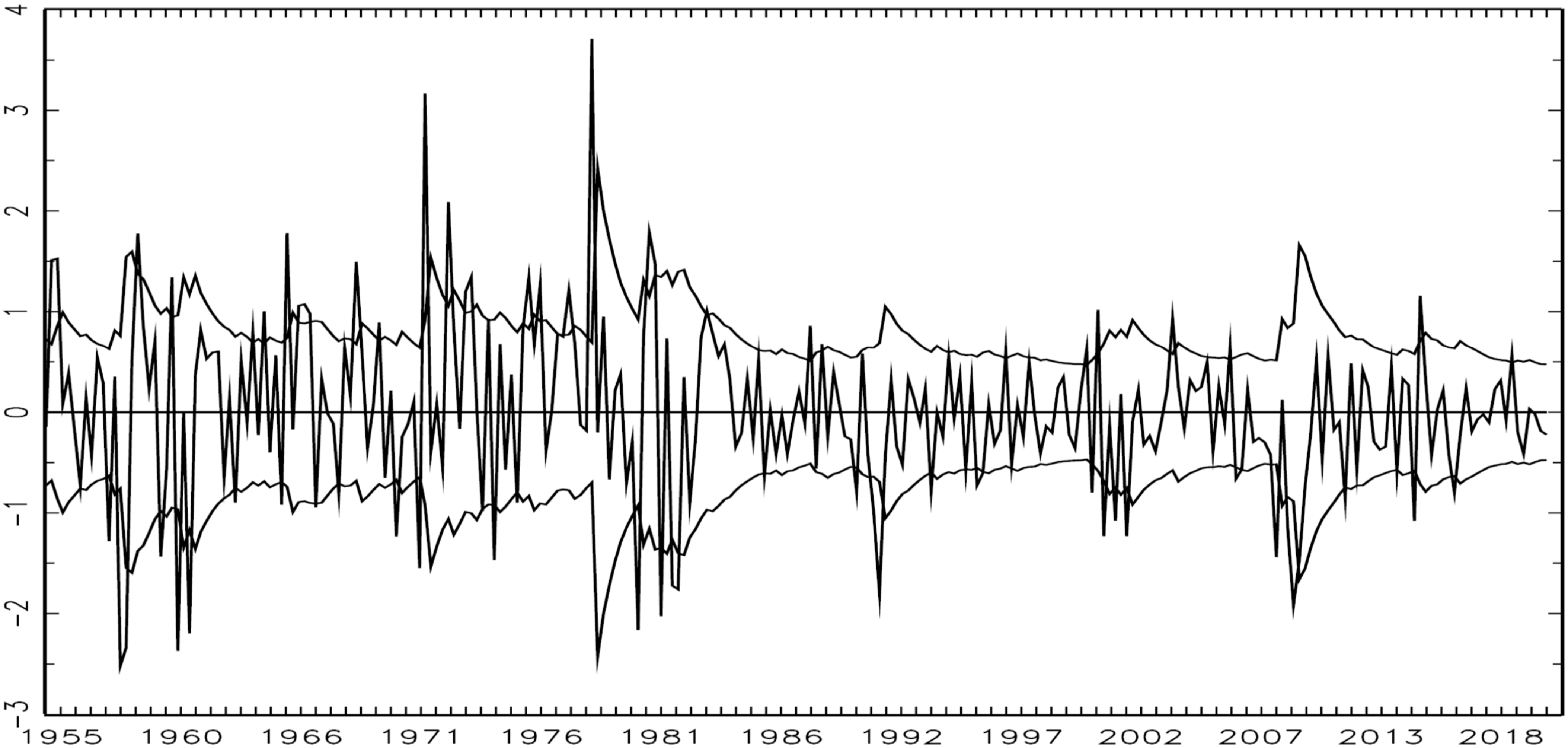
Heteroscedasticity

4(a) $\hat{\epsilon}_{z,t}$ and $\hat{\epsilon}_{z,t} \pm \hat{\sigma}_{z,t}$ (effective federal funds rate).



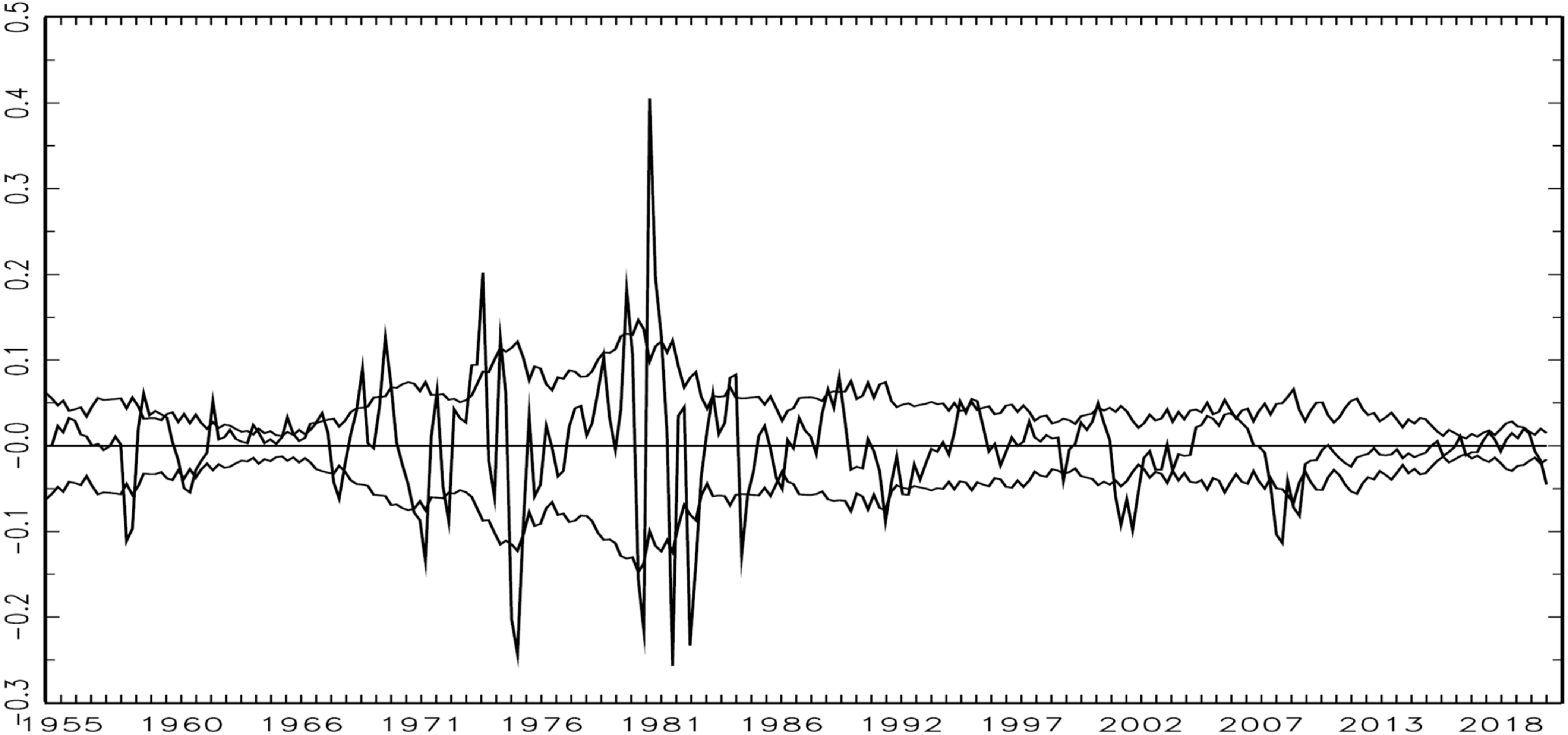
Heteroscedasticity

4(b) $\hat{\epsilon}_{g,t}$ and $\hat{\epsilon}_{g,t} \pm \hat{\sigma}_{g,t}$ (US GDP level).

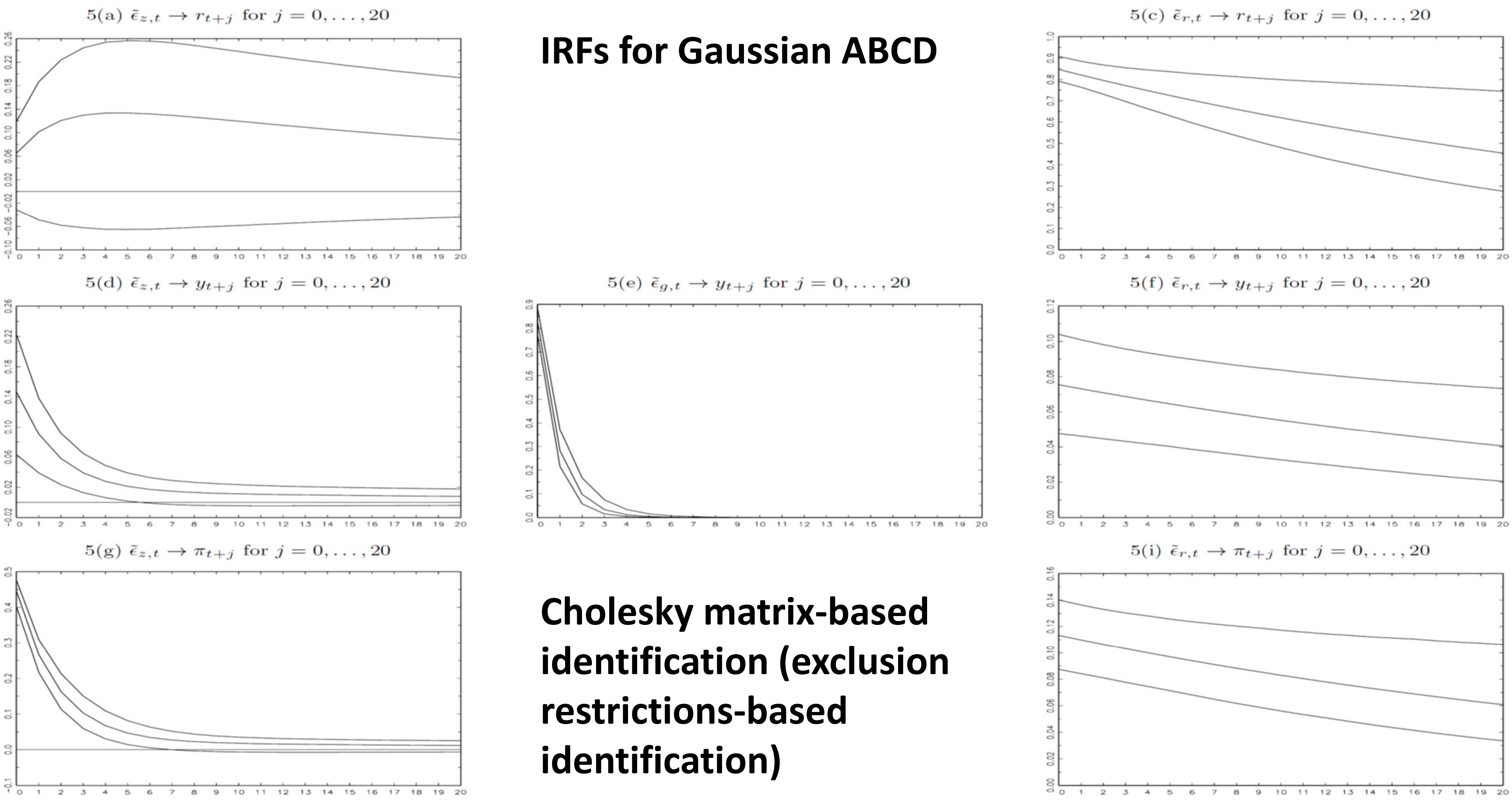


Heteroscedasticity

4(c) $\hat{\epsilon}_{r,t}$ and $\hat{\epsilon}_{r,t} \pm \hat{\sigma}_{r,t}$ (US inflation rate).



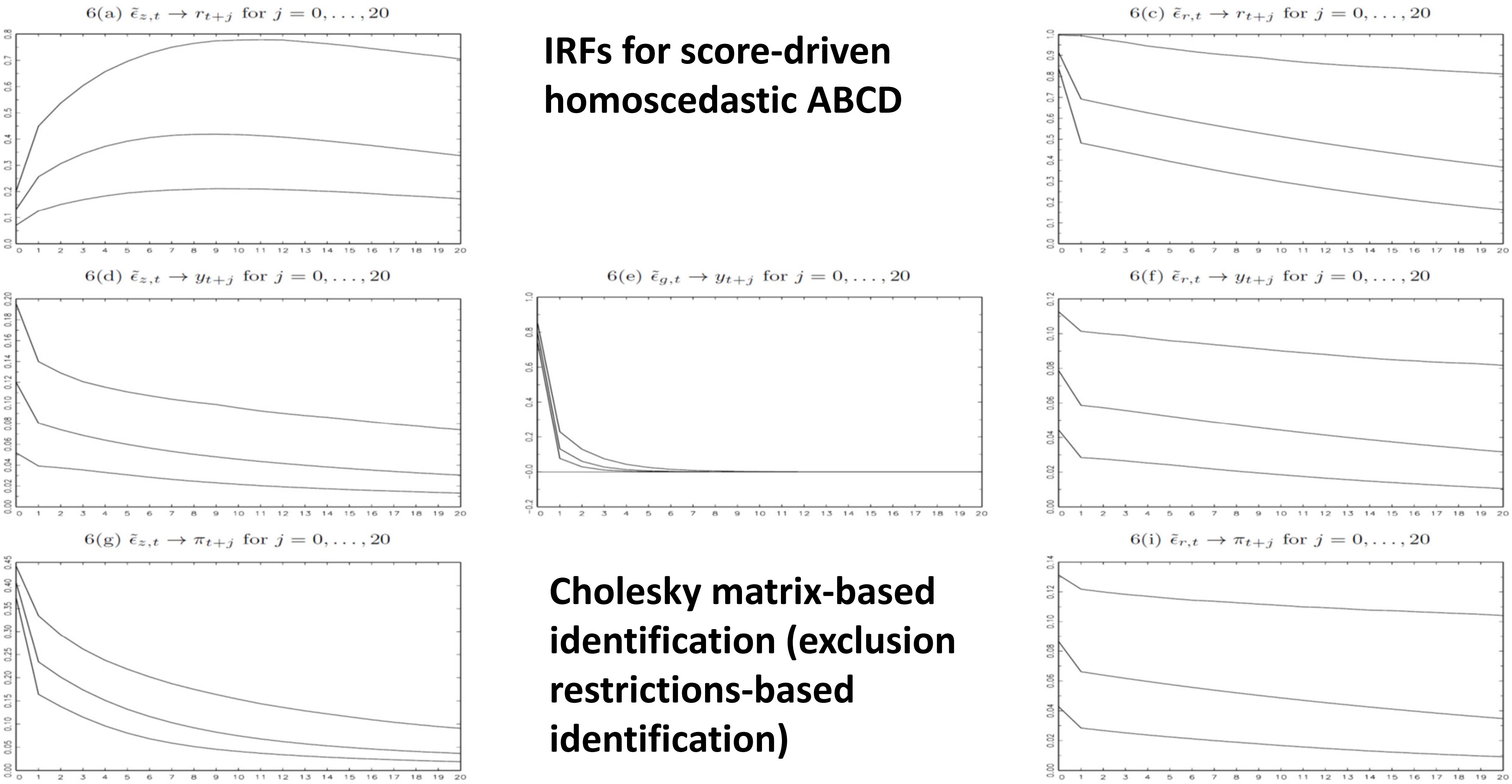
IRFs for Gaussian ABCD



Cholesky matrix-based identification (exclusion restrictions-based identification)

Figure 5. IRFs for Gaussian ABCD.

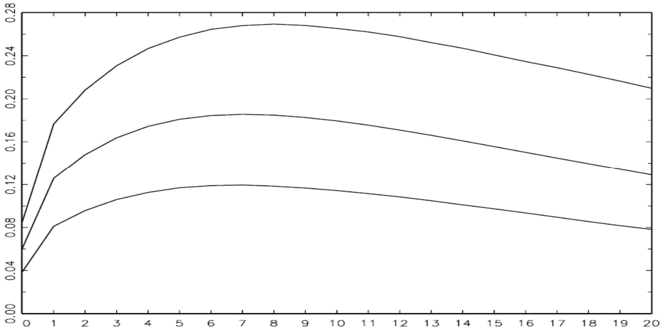
IRFs for score-driven homoscedastic ABCD



Cholesky matrix-based identification (exclusion restrictions-based identification)

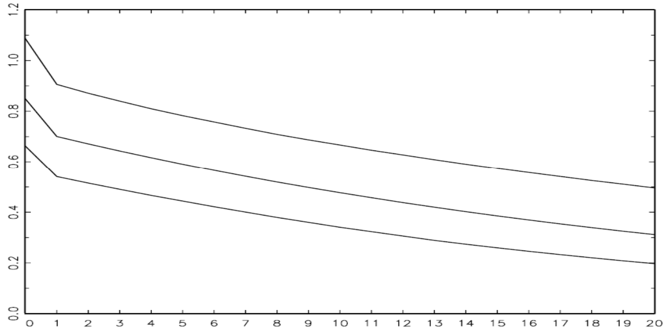
Figure 6. IRFs for score-driven homoskedastic ABCD.

7(a) $\tilde{\epsilon}_{z,t} \to r_{t+j}$ for $j = 0, \ldots, 20$

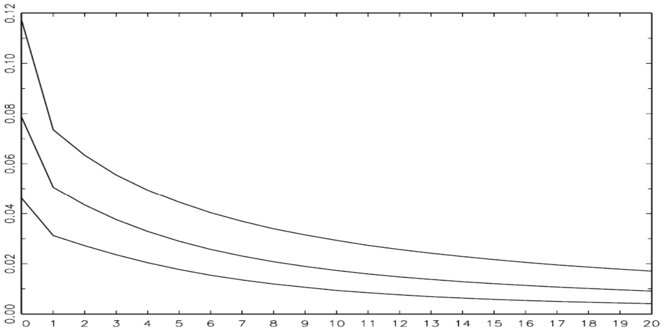


IRFs for score-driven heteroscedastic ABCD

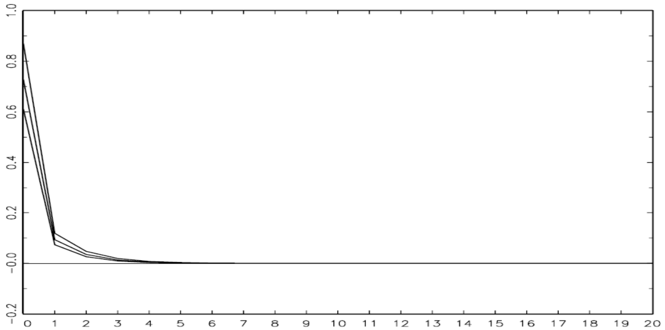
7(c) $\tilde{\epsilon}_{r,t} \to r_{t+j}$ for $j = 0, \ldots, 20$



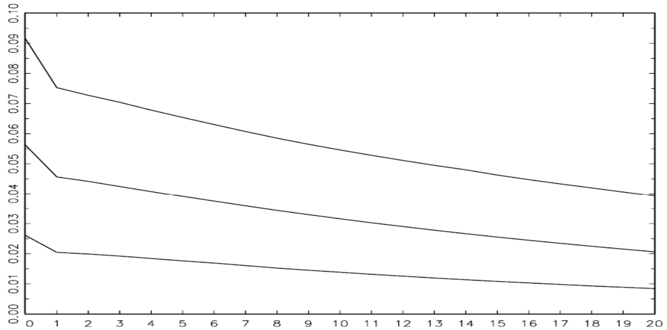
7(d) $\tilde{\epsilon}_{z,t} \to y_{t+j}$ for $j = 0, \ldots, 20$



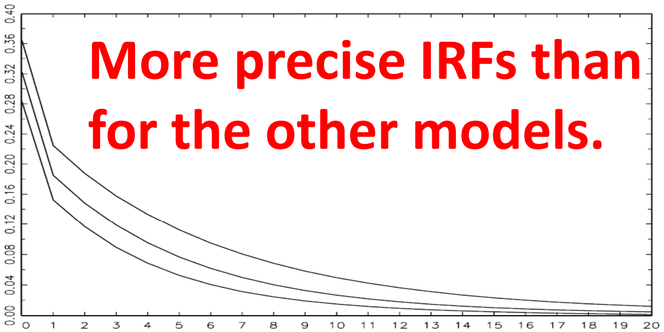
7(e) $\tilde{\epsilon}_{g,t} \to y_{t+j}$ for $j = 0, \ldots, 20$



7(f) $\tilde{\epsilon}_{r,t} \to y_{t+j}$ for $j = 0, \ldots, 20$



7(g) $\tilde{\epsilon}_{z,t} \to \pi_{t+j}$ for $j = 0, \ldots, 20$



More precise IRFs than for the other models.

Cholesky matrix-based identification (exclusion restrictions-based identification)

7(i) $\tilde{\epsilon}_{r,t} \to \pi_{t+j}$ for $j = 0, \ldots, 20$

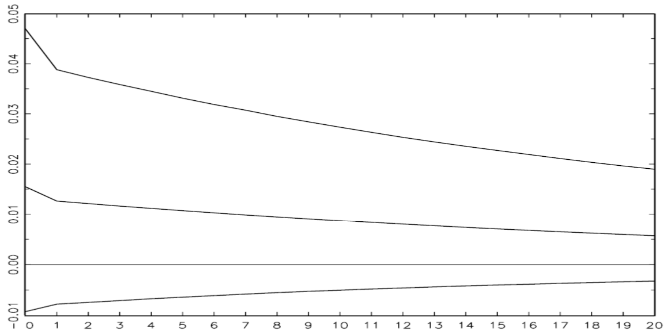


Figure 7. IRFs for score-driven heteroskedastic ABCD.

Thank you for your attention

sblazsek@ufm.edu

- An, Sungbae, and Frank Schorfheide. (2007). Bayesian Analysis of DSGE Models. **Econometric Reviews**.
- Angelini, G. and P. Gorgi (2018). DSGE models with observation-driven time-varying volatility. **Economics Letters**, 171, 169-171.
- Fernandez-Villaverde, Jesus, Juan F. Rubio-Ramirez, Thomas J. Sargent, and Mark W. Watson. (2007). ABCs (and Ds) of Understanding VARs. **American Economic Review**.
- Giacomini, Raffaella. (2013). The Relationship Between DSGE and VAR Models. In: VAR Models in Macroeconomics -- New Developments and Applications: Essays in Honor of Christopher A. Sims. Advances in Econometrics 32. Emerald Group Publishing Limited.
- Harvey, Andrew C., and Tirthankar Chakravarty. (2008). Beta-t-(E)GARCH. Cambridge Working Papers in Economics 0840.
- Iskrev, Nikolay. (2010). Local Identification in DSGE Models. **Journal of Monetary Economics**.
- Kociecki, Andrzej, and Marcin Kolasa. (2018). Global Identification of Linearized DSGE Models. **Quantitative Economics**.
- Komunjer, Ivana, and Serena Ng. (2011). Dynamic Identification of Dynamic Stochastic General Equilibrium Models. **Econometrica**.
- Morris, Steven D. (2014). Identification of Dynamic Stochastic General Equilibrium Models. Dissertation for Ph.D. in Economics, UC San Diego.